From Hexaflexagons to Edge Flexagons to Point Flexagons

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Edge flexagons

Hexaflexagons, the subject of Martin Gardner's column [**2**], were the first family of flexagons to be discovered. They are, however, only *one* example of an *edge flexagon*, a folded band of hinged, usually identical, convex polygons (called *leaves*). The strip of polygons used to construct a flexagon is called a *net*. In a *main position* an edge flexagon has the appearance of a ring of hinged polygons (see Figure 1). The rings are not necessarily flat and what look like single polygons are folded piles of one or more leaves, called *pats*. By definition, edge flexagons can be flexed to other main positions to display other pairs of faces, usually in a cyclic order. When flexing, leaf-bending is sometimes allowed, but in main positions the leaves are flat. The two preceding articles in this issue, [**2**] and [**4**], concern hexaflexagons (Figure 1a). Here we describe some non-hexagonal forms and say something about their classification.

(a) Trihexaflexagon (b) Square flexagon (c) Pentagonal flexagon

Figure 1. Main positions of some edge flexagons.

Workable paper models of flexagons are easy to make; however, the mathematics is complex and just how a flexagon works is not immediately obvious from casual examination of a paper model. Even square flexagons (Figure 1b), the second family of flexagons to be discovered, are far from simple. Martin Gardner wrote [**3**], "Stone and his friends spent considerable time folding and analyzing these four-sided forms,

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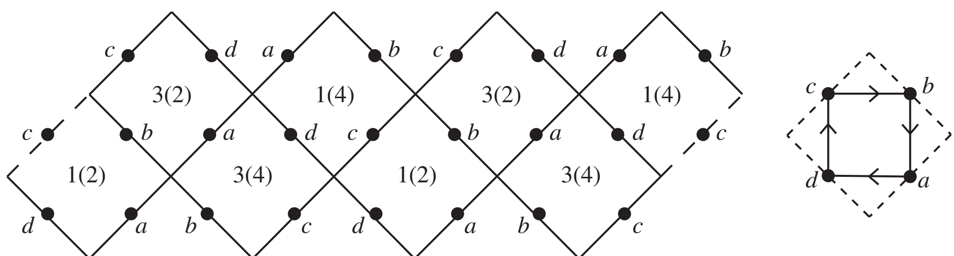
but did not succeed in developing a comprehensive theory that would cover all their discordant variations," a significant statement as Stone's friends included John Tukey and Richard Feynman.

Classification of edge flexagons

This is a difficult subject, as Martin Gardner's remark indicates. Certain edge flexagons, however, *can* be classified on the basis of their *characteristic polygon*, which describes how the hinge edges are linked by the leaves and is related to the vertex figures used to characterize polyhedra [**1**]. In particular, by describing how the leaves are linked, the characteristic polygon contains most of the information needed to construct the flexagon. For those edge flexagons that have a characteristic polygon, classification amounts to classification of all possible types of characteristic polygons.

We give two examples (in Figures 2 and 3) of edge flexagons and their characteristic polygons. To make paper models of any of the flexagons described here, use heavy paper stock and enlarge the nets to about twice the printed size. Transfer the numbers in parentheses from the upper side to the lower side and delete the numbers from the upper side.

In Figure 2a, we have the net for a square edge flexagon. To assemble this model, fold together pairs of leaves numbered 3 and 4, and join the ends (dashed lines). As assembled the flexagon is in main position (Figure 1b). This is flat and consists of alternate pats of single leaves and fan folded piles of three leaves. The flexagon can be flexed around a 4-cycle by folding it in two (to an intermediate position) and then unfolding into a new main position, and so on. The operation of this flexagon is similar to the traditional Chinese falling block toy sometimes called Jacob's Ladder [**8**].



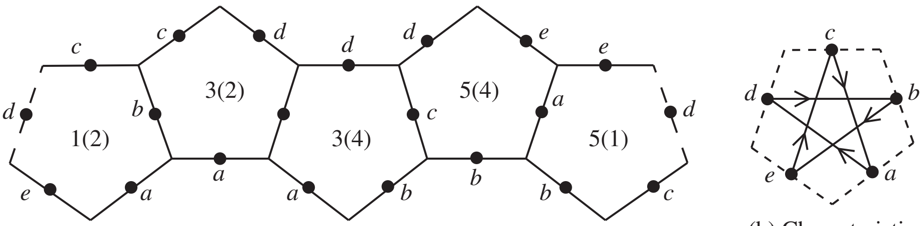
(a) Net

(b) Characteristic polygon

Figure 2. A square edge flexagon.

The characteristic polygon is obtained by joining the midpoints of the hinge edges on each leaf in the order they are linked by the leaves. This results in a twice-traced square (Figure 2b). Incidentally, joining end-to-end *two* copies of the net shown in Figure 2a results in the net for another square edge flexagon, the *octopus flexagon*. Main positions are *not* flat, but it can still be flexed around a 4-cycle using a pinch flex similar to the flex used for the trihexaflexagon. Other flexes are possible, and its overall behaviour is bewilderingly complex [**3**].

There are two regular pentagons: the usual convex one and the pentagram. This gives rise to two series of pentagonal flexagons, one of whose net is in Figure 3a. To assemble, first make *two* copies of Figure 3a, turn one over and join them end-to-end.



(a) Half the net

(b) Characteristic polygon

Figure 3. A pentagonal edge flexagon.

Next, fold together pairs of leaves numbered 2, 3, 4 and 5, and join the ends. Assembled, the flexagon is in an intermediate position which looks like a pair of pentagons with a common edge. From here, it can be opened into a type of main position (Figure 1c), and then closed into another main position in one continuous aesthetically satisfying movement, and so on around a 5-cycle. It can also be opened into another type of main position, but flexing round a 5-cycle is then complicated [**5**]. This is a difficult flexagon to handle.

Point flexagons

Point flexagons are the duals of edge flexagons. They are a recent discovery, first described by Scott Sherman in 2007 [**5**]. Nets are derived by truncating the leaves of the net of an edge flexagon at the vertices, so that vertices become edges and edges become vertices (see Figure 4). Edge hinges, therefore, become *point hinges*, usually modelled by short paper strips. By definition, a point hinge has only *two* degrees of freedom. The triangles in Figure 4 may rotate relative to each other about the y-axis, and the z-axis, but *not* be twisted relative to the x-axis.

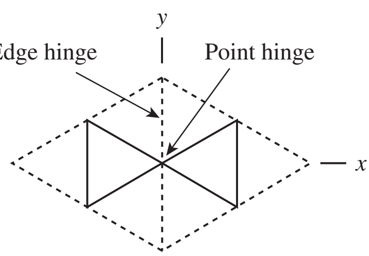
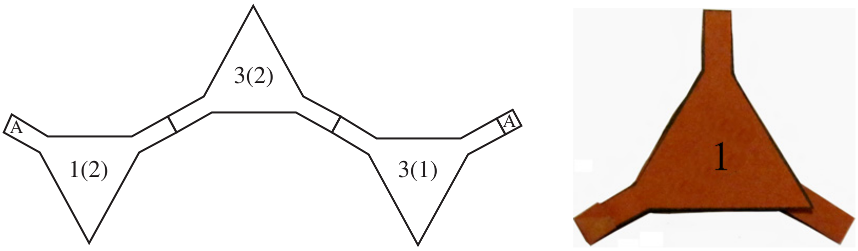


Figure 4. Truncating the edge-hinged triangles (dashed) creates point-hinged triangles (solid).

Figure 5a gives a net for a point flexagon with three faces. To assemble, crease the lines between the leaves to form hinges. Fold pairs of leaves numbered 2 and 3 together and glue the two part paper strips together at A-A. Assembled, the visible leaves, above and below, will both be numbered 1. The flexagon can be flexed around a 3-cycle by turning over the top leaf so that it becomes the bottom leaf, and so on, rather like the way cards are flipped in a Rolodex rotary file. This is analogous to the 3-cycle of the trihexaflexagon described by Martin Gardner [**2**]. The flex is not completely smooth since the hinge folds must be reversed and they are not parallel to the axis of rotation.



(a) Net (b) Assembled

Figure 5. A triangle point flexagon (dual to the trihexaflexagon).

Conclusion

Martin Gardner opened a Pandora's box when he introduced flexagons to a wide audience. Fifty years later, they are a mature topic with an extensive literature. They continue to fascinate with new results and new flexagons still being discovered and still to be discovered.

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Summary. Flexagons, introduced to a wide audience 50 years ago by Martin Gardner, now have an extensive literature and are an active field of research. This paper describes two kinds: edge flexagons and point flexagons, and gives an example of one means of classification.

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